Deformed Maxwell Algebras and their Realizations

Joaquim Gomis*, Kiyoshi Kamimura† and Jerzy Lukierski**

*Departament d'Estructura i Constituents de la Matèria and ICCUB, Universitat de Barcelona, Diagonal 647, 08028 Barcelona
†Department of Physics, Toho University, Funabashi, 274-8510 Japan
**Institute of Theoretical Physics, Wroclaw University, pl. Maxa Borna 9, 50-204 Wroclaw, Poland

Abstract. We study all possible deformations of the Maxwell algebra. In $D = d + 1 \neq 3$ dimensions there is only one-parameter deformation. The deformed algebra is isomorphic to $so(d+1,1) \oplus so(d,1)$ or to $so(d,2) \oplus so(d,1)$ depending on the signs of the deformation parameter. We construct in the $dS(AdS)$ space a model of massive particle interacting with Abelian vector field via non-local Lorentz force. In D=2+1 the deformations depend on two parameters b and k. We construct a phase diagram, with two parts of the (b,k) plane with $so(3,1) \oplus so(2,1)$ and $so(2,2) \oplus so(2,1)$ algebras separated by a critical curve along which the algebra is isomorphic to $Iso(2,1) \oplus so(2,1)$. We introduce in D=2+1 the Volkov-Akulov type model for a Abelian Goldstone-Nambu vector field described by a non-linear action containing as its bilinear term the free Chern-Simons Lagrangean.

INTRODUCTION

[1]

It is known since 1970 that the lagrangian of a massive relativistic particle in a constant electromagnetic background f^0_{ab} is invariant under a modification of Poincare algebra that leave the background invariant, (BCR algebra [2]). Its eight generators are two Lorentz transformations G, G^*, the four space-time translations and two central charges $Z, \tilde{Z}$ corresponding to the electric and magnetic charges. These central charges appear in the commutator of the four momenta. Later in 1972 Schrader [3], introduced the Maxwell algebra with 16 generators, the Poincare algebra plus six non-central extension

$$[P_a, \, P_b] = i \, Z_{ab}, \qquad Z_{ba} = -Z_{ab}. \tag{1}$$

---

[1] Talk based on [1] in the XXV-th Max Born Symposium "Planck Scale", held in Wroclaw 29.06-3.07.2009. , ed. R.Durka, J.Kowalski-Glikman and M.Szczochor, AIP Conference Proceedings Series, Melville(N.Y.).

The new generators Z_{ab} describe so called tensorial central charges[2] and satisfy the relations

$$[M_{ab}, Z_{cd}] = -i\,(\eta_{bc}\,Z_{ad} - \eta_{bd}\,Z_{ac} + \eta_{ad}\,Z_{bc} - \eta_{ac}\,Z_{bd}). \qquad (2)$$

As we discussed above a dynamical realization of BCR algebra is obtained by considering the relativistic particle coupled in minimal way to the electromagnetic potential $A_b = \frac{1}{2}f^0_{ab}x^a$ defining the constant field strength $F_{ab} = f^0_{ab}$. The second order lagrangian is

$$L = -m\sqrt{-\dot{x}^2} + \frac{e}{2}f^0_{ab}x^a\dot{x}^b. \qquad (3)$$

Note that this action is not invariant under the whole Maxwell algebra since part of the Lorentz rotations is broken by the choice of constant electromagnetic field strength f^0_{ab}. In order to recover the Maxwell symmetry one has to promote f^0_{ab} to be the dynamical degrees of freedom and consider an extension of space-time by supplementing the new coordinates $\theta^{ab}(=-\theta^{ba})$ which are the group parameters associated to Z_{ab}. In order to introduce the dynamics invariant under the Maxwell group symmetries we have applied in [10] [11] the method of non-linear realizations employing the Maurer Cartan (MC) one-forms (see e.g. [12],[13]).

The aim of this paper is to describe in any dimensions D all possible deformations of the Maxwell algebra (1), (2), and investigate the dynamics realizing the deformed Maxwell symmetries. In $D \neq 2+1$ there exists only one-parameter deformation which leads for positive (negative) value of the deformation parameter k to an algebra that is isomorphic to the direct sum of the AdS algebra $so(d,2)$ (dS algebra $so(d+1,1)$) and the Lorentz algebra $so(d,1)$. This deformation for $k > 0$ has been firstly obtained by Soroka and Soroka [14],[15].

In $D=2+1$ one gets a two-parameter family of deformations, with second deformation parameter b. The parameter space (b,k) is divided in two regions separated by the critical curve

$$A(b,k) = \left(\frac{k}{3}\right)^3 - \left(\frac{b}{2}\right)^2 = 0 \qquad (4)$$

on which the deformed algebra is non-semisimple. It appears that for $A > 0$ ($A < 0$) the deformed algebra is isomorphic to $so(2,2)\oplus so(2,1)$ ($so(3,1)\oplus so(2,1)$). On the curve (4) the deformed algebra is the direct sum of $D=2+1$ Poincare algebra and $D=2+1$ Lorentz algebra, $Iso(2,1)\oplus so(2,1)$.

In order to study the particle dynamics in the deformed cases we consider the MC one-forms on the suitable coset of deformed Maxwell group. Firstly we obtain, for arbitrary D and $k \neq 0, b = 0$, the particle model in curved and enlarged space-time $y^A = (x^a, \theta^{ab})$. We choose the coset which leads to the metric depending only on the space-time coordinates x^a. We derive in such a case the particle model in AdS (for $k > 0$) or dS (for $k < 0$) curved space-time with the coupling to Abelian vector field which generalizes, in the theory with deformed Maxwell symmetry, the Lorentz force

---

[2] We restrict ourselves to Minkowski space. Later in the literature the tensorial central charges were introduced mostly in the Poincare superalgebras [4] [5] [6] and also in p-brane non-relativistic Galilei and Newton-Hooke algebras [7] [8] [9].

term describing the particle interaction with constant electromagnetic field. The Lorentz force in the case studied here becomes non-local.

In $D=2+1$ and $k=0, b \neq 0$, we will consider a nonlinear field theory realization of the deformed Maxwell algebra in six-dimensional enlarged space $(x^a, \theta^a = \frac{1}{2}\varepsilon^{abc}\theta_{bc}; a, b = 0, 1, 2)$ by assuming that the surface $\theta^a = \theta^a(x)$ describes $D=2+1$ dimensional Goldstone vector fields[3]. If we postulate the action of Volkov-Akulov type [16] [17] we shall obtain the field theory in $D=2+1$ space-time with a lagrangian containing a free Abelian Chern-Simons term[18], [19].

The organization of the paper is as follows. After reviewing some properties of the Maxwell group we consider the corresponding particle model. Further we will present all possible deformations of Maxwell algebra. We construct also the deformed particle model for arbitrary D with $k \neq 0, b = 0$. For $D=2+1$ case with $k = 0, b \neq 0$ we promote the group parameters θ^a to Goldstone fields $\theta^a(x)$. These Goldstone-Nambu fields will be described by Volkov-Akulov type action. Finally we present a short summary.

PARTICLE MODEL FROM THE MAXWELL ALGEBRA (ARBITRARY D)

A particle model invariant under the complete Maxwell algebra can be derived geometrically [11] by the techniques of non-linear realizations, see e.g. [12]. Let us consider the coset G/H=Maxwell/Lorentz [10] [11] parametrized by $g = e^{iP_a x^a} e^{\frac{i}{2}Z_{ab}\theta^{ab}}$. The corresponding Maurer-Cartan (MC) one-forms are

$$\Omega = -ig^{-1}dg = P_a \, e^a + \frac{1}{2}Z_{ab} \, \omega^{ab} + \frac{1}{2}M_{ab} \, l^{ab}, \tag{5}$$

where

$$e^a = dx^a, \quad \omega^{ab} = d\theta^{ab} + \frac{1}{2}(x^a \, dx^b - x^b \, dx^a), \quad l^{ab} = 0. \tag{6}$$

A second order form of the lagrangian for the particle invariant under the full Maxwell algebra in the extended space-time $y^A = (x^a, \theta^{ab})$ can be written using the extra coordinates f_{ab} that transform covariantly under the Lorentz group [10]

$$\tilde{L} = -m\sqrt{-\dot{x}^2} + \frac{1}{2}f_{ab}\left(\dot{\theta}^{ab} + \frac{1}{2}(x^a\dot{x}^b - x^b\dot{x}^a)\right). \tag{7}$$

The Euler-Lagrange equations of motion are

$$\dot{f}_{ab} = 0, \quad \dot{\theta}^{ab} = -\frac{1}{2}(x^a\dot{x}^b - x^b\dot{x}^a), \quad m\frac{d}{d\tau}\frac{\dot{x}_a}{\sqrt{-\dot{x}^2}} = f_{ab}\dot{x}^b. \tag{8}$$

Integration of (8) gives $f_{ab} = f_{ab}^0$ and such a solution breaks the Lorentz symmetry spontaneously into the BCR subalgebra of Maxwell algebra. Substituting this solution

---

[3] Such a method was used firstly by Volkov and Akulov to derive the Goldstino field action [16].

in the *x*-equation of motion in (8) we provide the motion of a particle in the constant electromagnetic field [2],[3] described by the lagrangian (3). Notice that the interaction part of the lagrangian (7) defines an analogue of the EM potential $\hat{A}$ as one-form in the extended bosonic space (x, θ, f)

$$\hat{A} = \frac{1}{2} f_{ab} \omega^{ab}, \qquad with \qquad \hat{F} = d\hat{A} = \frac{1}{2} f_{ab} e^a \wedge e^b + \frac{1}{2} df_{ab} \wedge \omega^{ab}. \qquad (9)$$

We see that the field strength has the constant components f_{ab} on-shell, $\dot{f}_{ab} = 0$.

The infinitesimal symmetries of the lagrangian (7) are realized canonically as Noether generators

$$\begin{aligned}
\mathscr{P}_a &= -\left(p_a - \frac{1}{2} p_{ab} x^b \right), \qquad \mathscr{Z}_{ab} = -p_{ab}, \\
\mathscr{M}_{ab} &= -\left(p_{[a} x_{b]} + p_{[ac} \theta_{b]}{}^c + p^f_{[ac} f_{b]}{}^c \right),
\end{aligned} \qquad (10)$$

where p_a, p_{ab}, p_f^{ab} are the canonically conjugated momenta of the coordinates x^a, θ^{ab}, f_{ab}. From the lagrangian (7) we obtain the constraints

$$\phi = \frac{1}{2} \left(\pi_a^2 + m^2 \right) = 0, \qquad \phi_{ab} = p_{ab} - f_{ab} = 0, \qquad \phi_f^{ab} = p_f^{ab} = 0, \qquad (11)$$

where $\pi_a \equiv p_a + \frac{1}{2} f_{ab} x^b$. The last two are the second class constraints and are solved as $(f_{ab}, p_f^{ab}) = (p_{ab}, 0)$. The Hamiltonian becomes

$$\mathscr{H} = \lambda \, \phi = \frac{\lambda}{2} \left(\pi_a^2 + m^2 \right) \qquad (12)$$

and the constraints (11) and the global generators (10) are shown to be conserved.

There are four Casimirs in the Maxwell algebra in four dimensions, [3],[14]

$$\begin{aligned}
C_1 &= \mathscr{P}_a^2 - \mathscr{M}_{ab} \mathscr{Z}^{ab}, \qquad C_2 = \frac{1}{2} \mathscr{Z}_{ab}^2, \\
C_3 &= (\mathscr{Z} \tilde{\mathscr{Z}}), \qquad C_4 = (\mathscr{P}^b \tilde{\mathscr{Z}}_{ba})^2 + \frac{1}{4} (\mathscr{Z} \tilde{\mathscr{Z}}) (\mathscr{M}_{ab} \tilde{\mathscr{Z}}^{ab}),
\end{aligned} \qquad (13)$$

where $\tilde{\mathscr{Z}}^{ab} = \frac{1}{2} \varepsilon^{abcd} \mathscr{Z}_{cd}$. In the first quantized theory they are imposed in the Schrödinger representation. In particular the first one gives the generalized KG equation,

$$\left[\left(\frac{1}{i} \frac{\partial}{\partial x^a} + \frac{1}{2i} x^b \frac{\partial}{\partial \theta^{ab}} \right)^2 + m^2 \right] \Psi(x^a, \theta^{ab}) = 0. \qquad (14)$$

DEFORMATIONS OF MAXWELL ALGEBRA

In this section we would like to find all possible deformations of the Maxwell algebra. The problem of finding the continuous deformations of a Lie algebra can be described in cohomological terms [20], we follow the notations of ref.[21].

The MC form for the Maxwell algebra is

$$\Omega = P_a L_P^a + \frac{1}{2} Z_{ab} L_Z^{ab} + \frac{1}{2} M_{ab} L_M^{ab} \tag{15}$$

and the MC equations in this case are given by[4]

$$\begin{aligned}
dL_M^{ab} + L_M^{ac} L_{Mc}{}^b &= 0, \\
dL_P^a + L_M^{ac} L_{Mc} &= 0, \\
dL_Z^{ab} + L_Z^{ac} L_{Mc}{}^b + L_M^{ac} L_{Zc}{}^b - L_P^a L_P^b &= 0.
\end{aligned} \tag{16}$$

The infinitesimal deformations are characterized by the non-trivial vector-valued two-forms $A^{(2)}$ verifying

$$DA^{(2)} = 0, \qquad A^{(2)} \neq -D\Phi^{(1)}. \tag{17}$$

The non-trivial infinitesimal deformations are in one to one correspondence with the second cohomology group $H^2(g;g)$. In the case when $H^3(g;g)$ vanishes, it is always possible to choose a representative in the class of infinitesimal deformations such that it verifies the Jacobi identity in all orders.

Solving (17) we find a one-parameter family of non-trivial solutions for $A^{(2)}$, with the exception that there is a two-parameter family in "exotic" case $D=2+1$.[5] The MC equations get additional terms representing deformations as follows

$$\begin{aligned}
dL_M^{ab} + L_M^{ac} L_{Mc}{}^b &= b\,\varepsilon^{abc} L_{Zcd} L_P^d, \\
dL_P^a + L_M^{ac} L_{Pc} &= k L_Z^{ac} L_{Pc} + b\,\frac{1}{4} L_Z^{ab} \varepsilon_{bcd} L_Z^{cd}, \\
dL_Z^{ab} + L_Z^{ac} L_{Mc}{}^b + L_M^{ac} L_{Zc}{}^b - L_P^a L_P^b &= k L_Z^{ac} L_{Zc}{}^b, \qquad (\varepsilon^{012} = -\varepsilon_{012} = 1). \tag{18}
\end{aligned}$$

Here k and b are arbitrary real constant parameters; we stress that deformation terms proportional to b are present only in $D=2+1$. The length dimensions of k and b are respectively $[L^{-2}]$ and $[L^{-3}]$.

The general deformed Maxwell algebra found in the previous subsection can be written in terms of the commutators of generators. In general dimensions there exists only the following k-deformed algebra, with $b=0$

$$\begin{aligned}
[P_a, P_b] &= i\, Z_{ab}, & [M_{ab}, M_{cd}] &= -i\,\eta_{b[c}\, M_{ad]} + i\,\eta_{a[c}\, M_{bd]}, \\
[P_a, M_{bc}] &= -i\,\eta_{a[b}\, P_{c]}, & [Z_{ab}, M_{cd}] &= -i\left(\eta_{b[c}\, Z_{ad]} - \eta_{a[c}\, Z_{bd]}\right), \\
[P_a, Z_{bc}] &= +ik\,\eta_{a[b}\, P_{c]}, & [Z_{ab}, Z_{cd}] &= +ik\left(\eta_{b[c}\, Z_{ad]} - i\,\eta_{a[c}\, Z_{bd]}\right). \tag{19}
\end{aligned}$$

---

[4] As usual we will often omit "∧" for exterior product of forms.
[5] Some of the calculations with forms are being done using the Mathematica code for differential forms EDC [22].

For $k \neq 0$ case we introduce dimensionless rescaled generators as $\mathscr{P}_a = \frac{P_a}{\sqrt{|k|}}, \mathscr{M}_{ab} = -\frac{Z_{ab}}{k}, \mathscr{J}_{ab} = M_{ab} + \frac{Z_{ab}}{k}$, then the k-deformation of Maxwell algebra becomes

$$
\begin{aligned}
[\mathscr{P}_a, \mathscr{P}_b] &= -i\frac{k}{|k|}\,\mathscr{M}_{ab}, \\
[\mathscr{P}_a, \mathscr{M}_{bc}] &= -i\,\eta_{a[b}\,\mathscr{P}_{c]}, \qquad [\mathscr{M}_{ab}, \mathscr{M}_{cd}] = -i\,\eta_{b[c}\,\mathscr{M}_{ad]} + i\,\eta_{a[c}\,\mathscr{M}_{bd]}, \\
[\mathscr{P}_a, \mathscr{J}_{bc}] &= [\mathscr{M}_{ab}, \mathscr{J}_{cd}] = 0, \quad [\mathscr{J}_{ab}, \mathscr{J}_{cd}] = -i\,\eta_{b[c}\,\mathscr{J}_{ad]} + i\,\eta_{a[c}\,\mathscr{J}_{bd]}. \quad (20)
\end{aligned}
$$

The algebra of $(\mathscr{P}_a, \mathscr{M}_{cd}, \mathscr{J}_{cd},)$ for $k > 0$ (k^+-deformation) is $so(D-1,2) \oplus so(D-1,1)$, i.e. we obtain the direct sum of AdS_D and D-dimensional Lorentz group. For $k < 0$ (k^--deformation) we get $so(D,1) \oplus so(D-1,1)$, i.e., the direct sum of dS_D and D-dimensional Lorentz group. We recall here that the above algebra for $k > 0$ was previously found by Soroka and Soroka [15]. In our further discussion we will also use the notation $k = \pm\frac{1}{R^2}$ where R is the radius of AdS ($k > 0$) or dS ($k < 0$) space.

Three dimensional case is interesting since there is an exotic b-deformation of the Maxwell algebra in addition to the k-deformation. For $b = 0$, $k \neq 0$, as was discussed previously, the algebra is $so(2,2) \oplus so(2,1)$ for $k > 0$ (k^+-deformation) and $so(3,1) \oplus so(2,1)$ for $k < 0$ (k^--deformation). For $k = 0$, $b \neq 0$ (b-deformation) we can introduce

$$
\begin{pmatrix} \mathscr{P}_a \\ \mathscr{M}_a \\ \mathscr{J}_a \end{pmatrix} = \begin{pmatrix} & U_b & \end{pmatrix} \begin{pmatrix} P_a \\ M_a \\ Z_a \end{pmatrix}, \qquad U_b = \begin{pmatrix} \frac{1}{\sqrt{3}\,b^{1/3}} & 0 & \frac{1}{\sqrt{3}\,b^{2/3}} \\ -\frac{1}{3b^{1/3}} & \frac{2}{3} & \frac{1}{3b^{2/3}} \\ \frac{1}{3b^{1/3}} & \frac{1}{3} & -\frac{1}{3b^{2/3}} \end{pmatrix}, \quad (21)
$$

where $M^a = \frac{1}{2}\varepsilon^{abc}M_{bc}, Z^a = \frac{1}{2}\varepsilon^{abc}Z_{bc}$. The algebra becomes

$$
\begin{aligned}
[\mathscr{P}_a, \mathscr{P}_b] &= -i\varepsilon_{abc}\mathscr{M}^c, \quad [\mathscr{P}_a, \mathscr{M}_b] = i\varepsilon_{abc}\mathscr{P}^c, \quad [\mathscr{M}_a, \mathscr{M}_b] = i\varepsilon_{abc}\mathscr{M}^c, \\
[\mathscr{J}_a, \mathscr{J}_b] &= i\varepsilon_{abc}\mathscr{J}^c, \quad [\mathscr{P}_a, \mathscr{J}_b] = [\mathscr{M}_a, \mathscr{J}_b] = 0. \quad (22)
\end{aligned}
$$

Then $(\mathscr{P}_a, \mathscr{M}_a)$ are the $so(3,1)$ generators and $\mathscr{J}_a$ describes $so(2,1)$. This algebra is isomorphic to the one with $b = 0, k < 0$ (k^--deformation).

To examine more general case with any values of the deformation parameters (b,k) we consider the Killing form $g_{ij} = C_{ik}^{\ell}C_{\ell j}^{k}$ of the algebra. Its determinant is

$$
\det g_{ij} = 6^9 4^3 A(b,k)^3, \qquad A(b,k) \equiv (\frac{k}{3})^3 - (\frac{b}{2})^2. \quad (23)
$$

In the case $\det g = 0$ the Killing form is degenerate, otherwise the algebra is semisimple. In table 1 we summarize the cases accrding to the values of (k,b).

I	$\det g = 0$	$b = 0, k = 0,$	Maxwell	Maxwell algebra
II	$\det g = 0$	$A(b,k) = 0$	Poincaré	$Iso(2,1) \oplus so(2,1)$
III	$\det g > 0$	$A(b,k) > 0$	AdS	$so(2,2) \oplus so(2,1)$
IV	$\det g < 0$	$A(b,k) < 0$	dS	$so(3,1) \oplus so(2,1)$

Table 1: *The phase sectors for deformed D=2+1 Maxwell algebra*

PARTICLE MODELS ON THE k-DEFORMED MAXWELL ALGEBRA

In this section we will discuss a model realizing in arbitrary dimension D the deformed Maxwell algebras and look for the physical meaning of the additional coordinates (f_{ab}, θ^{ab}). Generalizing the results described earlier for the standard Maxwell algebra to those for the deformed Maxwell algebra with $(k \neq 0, b = 0)$, we describe a particle interacting with constant electromagnetic field in a generalized AdS(dS) space-time.

We consider a coset G/H with $G = \{P_a, M_{ab}, Z_{ab}\}, H = \{M_{ab}\}$ and parametrize the group element g using (x^a, θ^{ab}) as $g = e^{iP_a x^a} e^{\frac{i}{2} Z_{ab} \theta^{ab}}$. The MC form for Ω this coset is

$$
L_P^a \;=\; e^b \Lambda_b{}^a, \quad L_Z^{cd} = -\frac{1}{k} \Lambda^{-1c}{}_a \left(\omega^{ab} - (\Lambda d\Lambda^{-1})^{ab} \right) \Lambda_b{}^d, \quad L_M^{cd} = 0, \quad (24)
$$

where (e^a, ω^{ab}) are AdS(dS) drei-bein and spin connection satisfying

$$
de^a + \omega^a{}_b e^b = 0, \qquad d\omega^{ab} + \omega^a{}_c \omega^{cb} = -k e^a e^b \qquad (25)
$$

and Λ is a vector Lorentz transformations (Lorentz harmonics) in terms of new tensorial coordinates θ^{ab} as follows

$$
\Lambda_a{}^b = (e^{-k\theta})_a{}^b = \delta_a{}^b + (-k\theta)_a{}^b + \frac{1}{2!}(-k\theta)_a{}^c(-k\theta)_c{}^b + \dots. \qquad (26)
$$

The particle action generalizing (7) for $k \neq 0$ looks as follows

$$
\mathscr{L} d\tau = -m\sqrt{-\eta_{ab} L_P^{a*} L_P^{b*}} + \frac{1}{2} f_{ab} L_Z^{ab*} = -m\sqrt{-g_{ab}(x) \dot{x}^a \dot{x}^b} \, d\tau + \hat{A}^*, \qquad (27)
$$

where g_{ab} is the AdS(dS) metric, now depending only on x,

$$
g_{ab} \;=\; e_a{}^c e_b{}^d \eta_{cd} = \eta_{ab} + \left[\left(\frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}} \right)^2 - 1 \right] \left(\eta_{ab} - \frac{x_a x_b}{x^2} \right). \qquad (28)
$$

We obtain the metric of AdS (dS) space with radius R, where $k = 1/R^2$. The pullback $\hat{A}^*$ in (27) takes the explicit form

$$
\hat{A}^* \;=\; -\frac{1}{2} f_{ab} L_Z^{ba*} = -tr(\frac{1}{2} f L_Z^*) = +\frac{1}{2k} tr \left[f \Lambda^{-1} \left(\omega_\tau - \Lambda \partial_\tau \Lambda^{-1} \right) \Lambda \right] d\tau,
$$

$$
\omega_\tau^{cd} = \frac{\dot{x}^{[c} x^{d]}}{x^2} \left(\cos(\sqrt{kr^2}) - 1 \right). \qquad (29)
$$

In a limit $k \to 0$ (equivalently $R \to \infty$) we obtain the undeformed Maxwell case (9) [11].

Now we shall describe the equations of motion following from the lagrangian (27). They are (for details see [1])

$$
\omega_\tau - \Lambda \partial_\tau \Lambda^{-1} = 0, \qquad \dot{f}_{ab} = 0, \qquad m \nabla_\tau^2 x_a = F_{ab} \dot{x}^b, \qquad (30)
$$

where ∇_τ^2 is covariant second derivative. The field strength is

$$F_{ab}(x, \theta) \;=\; \tilde{f}_{ab}\left(\frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}}\right)^2 - \frac{\tilde{f}_{[ac}x^c x_{b]}}{x^2}\frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}}\left(\frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}} - 1\right), \quad (31)$$

where $\tilde{f}_{ab} = (\Lambda f \Lambda^{-1})_{ab}$. We see that for $k \neq 0$ the generalized Lorentz force depends on θ^{cd} but in the limit $k \to 0$ we get $F_{ab} = f_{ab}$ as expected. If we use (30) to express Λ in the terms of the coordinates x^a, we obtain a non-local Lorentz force.

b-DEFORMED MAXWELL ALGEBRA IN D=2+1 AND GOLDSTONE-NAMBU VECTOR FIELDS

Let us consider now the deformation $k = 0, b \neq 0$ in D=2+1. As an application of the *b*-deformed Maxwell algebra we discuss the Volkov-Akulov formula[23],[24] for invariant D=2+1 Goldstone field action,

$$S = \int \left(-\frac{1}{3!}\right)\varepsilon_{abc}L_P^{a\star}L_P^{b\star}L_P^{c\star} = \int d^3x\,\mathscr{L}_\theta, \qquad \mathscr{L}_\theta = \det(\tilde{e}^a{}_b), \qquad (32)$$

where $L_P^{a\star}$ is the pullback with respect to $x^a \to \theta^a$, then $d\theta^{a\star} = \frac{\partial\theta^a(x)}{\partial x^b}dx^b$; in such a way the Lagrangian density is a function of $\theta(x)$ and $\partial_a\theta(x)$. The MC forms are obtained as in the previous cases. Using the detailed expressions in [1] we obtain, for the small b,

$$\begin{aligned}
\mathscr{L}_\theta \;=\;& \det(\tilde{e}^a{}_b) = 1 - b\left((x\theta) + \frac{1}{2}\varepsilon_{abc}\theta^a\frac{\partial\theta^c}{\partial x_b}\right) + b^2\left((-\frac{3(x^2)^3}{2240} + \frac{x^2\,\theta^2}{12} + \frac{(x\theta)^2}{3})\right. \\
&-\; \frac{x^2}{48}((x\theta)\delta_j{}^i - x_j\theta^i)\frac{\partial\theta^j}{\partial x^i} + \frac{(x\theta)}{4}\varepsilon_{abc}\theta^a\frac{\partial\theta^c}{\partial x_b} + \frac{1}{8}\left.\varepsilon_{abc}\varepsilon^{def}\theta^a\theta_d\frac{\partial\theta^b}{\partial x^e}\frac{\partial\theta^c}{\partial x^f}\right) + \mathcal{O}(b^3).
\end{aligned}$$
$$(33)$$

The lagrangian density (33) contains as one of two terms linear in b the exact topological lagrangian for D=2+1 Chern-Simons field

$$\mathscr{L}^{CS} \;=\; -\frac{b}{2}\varepsilon_{abc}\theta^a\frac{\partial\theta^b}{\partial x_c}. \qquad (34)$$

If we consider higher order terms in b they can be treated as describing new interaction vertices implying that the Nambu-Goldstone field $\theta^a(x)$ looses its topological nature. The appearance of the terms depending explicitly on x^a and θ^a in (33) is related with the curved geometry in the extended space (x^a, θ^a).

OUTLOOK

In this paper we have reviewed the deformations of the Maxwell algebra studied in [1]. The general mathematical techniques permit us to solve the problem of complete

classification of these deformations. The commuting generators Z_{ab} in (1) become non-abelian in arbitrary dimension D and they are promoted to the $\frac{D(D-1)}{2}$ generators of $so(D-1,1)$ Lorentz algebra. The particle dynamics in the $\frac{D(D+1)}{2}$ dimensional coset with generators (P_a, Z_{ab}) becomes the theory of point particles moving on AdS (for $k > 0$) or dS (for $k < 0$) group manifolds in external electromagnetic fields. If we use standard formula (27) for the particle action in curved space-time one can show that the particle moves only in the space-time sector $(x^a, \theta^{ab} = 0)$ of the extended space-time (x^a, θ^{ab}) with a non-local Lorentz force.

In "exotic" dimension D=2+1 the symmetry corresponding to the two parameter deformation of Maxwell algebra is less transparent. The coset with generators (P_a, Z^a) and coordinates (x^a, θ_a) in D=2+1 if $b \neq 0$ is neither the group manifold nor even the symmetric coset space. In order to find the dynamical realization we assume that the coordinates x^a are primary and the coordinates θ^a describe the Goldstone field values. We derive a non-linear lagrangian for vector Goldstone field containing the bi-linear kinetic term describing exactly the D=2+1 CS Abelian action.

Finally we would like to point out some issues which deserve further investigation:

1) It is interesting to consider the supersymmetric extension of the BCR algebra, describing the symmetries of the space-time in the presence of constant electromagnetic field background, to the supersymmetries of superspace in general backgrounds of SUSY gauge fields. Further question is the formulation of the SUSY extension of the Maxwell algebra. These issues are under consideration[25].

2) As we already mentioned, the deformation parameter k with the dimensionality $[L^{-2}]$ can be described by the formula $|k| = \frac{1}{R^2}$, and interpreted as the AdS(dS) radius for $k > 0 (k < 0)$. The parameter b, with the dimensionality $[L^{-3}]$, if $k = 0$ is related with the closure of the quadrilinear relation for the following non Abelian translation generators P_a,

$$[[P_a, P_b], [P_c, P_d]] = ib \left(\eta_{a[c} \varepsilon_{bd]e} - \eta_{b[c} \varepsilon_{ad]e} \right) P^e. \tag{35}$$

This relation is an example of higher order Lie algebra for $n = 4$ [26, 27]. It is an interesting task to understand the translations (35) as describing some D=2+1 dimensional curved manifold.

3) Recently in [10][11] they were considered infinite sequential extensions of the Maxwell algebra with additional tensorial generators. The concrete form of these extensions can be determined by studying the Chevalley-Eilenberg cohomologies at degree two. The point particle models related with these Poincare algebra extensions have been studied in [10]. There appears an interesting question of the dynamical and physical interpretation of the additional tensorial degrees of freedom.

ACKNOWLEDGMENTS

REFERENCES

1. J. Gomis, K. Kamimura and J. Lukierski, JHEP **0908**, 039 (2009) [arXiv:0906.4464].
2. H. Bacry, P. Combe and J. L. Richard, *Nuovo Cim. A* **67** (1970) 267.
3. R. Schrader, Fortsch. Phys. **20** (1972) 701.
4. R. D'Auria and P. Fre, Nucl. Phys. B **201** (1982) 101 [Erratum-ibid. B **206** (1982) 496].
5. J. W. van Holten and A. Van Proeyen, J. Phys. A **15**, 3763 (1982).
6. J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo and P. K. Townsend, Phys. Rev. Lett. **63** (1989) 2443.
7. J. Brugues, T. Curtright, J. Gomis and L. Mezincescu, Phys. Lett. B **594** (2004) 227 [arXiv:hep-th/0404175].
8. J. Gomis, J. Gomis and K. Kamimura, JHEP **0512** (2005) 024 [arXiv:hep-th/0507036].
9. J. Brugues, J. Gomis and K. Kamimura, Phys. Rev. D **73** (2006) 085011 [arXiv:hep-th/0603023].
10. S. Bonanos and J. Gomis, J. Phys. A: Math. Theor. **42** (2009) 145206 [arXiv:hep-th/0808.2243].
11. S. Bonanos and J. Gomis, [arXiv:0812.4140].
12. S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. **177** (1969) 2239;
    C. G. . Callan, S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. **177** (1969) 2247.
13. J. Gomis, K. Kamimura and P. C. West, Class. Quant. Grav. **23** (2006) 7369 [arXiv:hep-th/0607057].
14. D. V. Soroka and V. A. Soroka, Phys. Lett. B **607** (2005) 302 [arXiv:hep-th/0410012].
15. D. V. Soroka and V. A. Soroka, [arXiv:hep-th/0605251].
16. D. V. Volkov and V. P. Akulov, Phys. Lett. B **46** (1973) 109.
17. B. Zumino, Nucl. Phys. B **127** (1977) 189.
18. A. S. Schwarz, Lett. Math. Phys. **2** (1978) 247.
19. S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48** (1982) 975.
20. M. Levy-Nahas, J. Math. Phys. **8**, 1211 (1967).
21. G. W. Gibbons, J. Gomis and C. N. Pope, Phys. Rev. D **76** (2007) 081701 [arXiv:0707.2174].
22. S. Bonanos, "Exterior Differential Calculus", Mathematica Enhancement, http: // www. inp. demokritos.gr/ sbonano/EDC/
23. D. V. Volkov, Fiz. Elem. Chast. Atom. Yadra **4** (1973) 3.
24. E. A. Ivanov and V. I. Ogievetsky, Teor. Mat. Fiz. **25** (1975) 164.
25. S. Bonanos, J. Gomis, K. Kamimura and J. Lukierski, in preparation.
26. P. Hanlon and M. Wachs, Advances in Mathematics, **113**, 206 (1995).
27. J. A. De Azcarraga and J. C. Perez Bueno, Commun. Math. Phys. **184**, 669 (1997) [arXiv:hep-th/9605213].